ON FIELDS OF DEFINITION OF ARITHMETIC KLEINIAN REFLECTION GROUPS. II

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ABSTRACT. Following the previous work of Nikulin and Agol, Belolipetsky, Storm, and Whyte it is known that there exist only finitely many (totally real) number fields that can serve as fields of definition of arithmetic hyperbolic reflection groups. We prove a new bound on the degree n_k of these fields in dimension 3: n_k does not exceed 9. Combined with previous results of Maclachlan and Nikulin, this leads to a new bound $n_k \leq 25$ which is valid for all dimensions. We also obtain upper bounds for the discriminants of these fields and give some heuristic results which may be useful for the classification of arithmetic hyperbolic reflection groups.

1. INTRODUCTION

A group of isometries of hyperbolic n-space $\mathbb{H}^n$ is called a *reflection group* if it has a finite generating set which consists of reflections in hyperplanes. The study of hyperbolic reflection groups has a long and remarkable history going back to the papers of Makarov and Vinberg. In recent years there has been a wave of activity in this area which has led to a solution to the open question of the finiteness of these groups and to some quantitative results towards their classification. An improvement of the quantitative bounds is the subject of the present paper.

Recall that a reflection group is called *maximal* if it is not properly contained in any other reflection group. In many cases, while studying hyperbolic reflection groups it is useful to restrict one's attention to *arithmetic groups* of isometries (see definition in Section 2). Answering a long standing open question, it was proven independently in [2] and [22] that there are only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups. This implies, at least theoretically, that these groups can be classified. Considering the ubiquity of hyperbolic reflection groups, their classification stands as a fundamental open problem. Our current knowledge falls short of a solution to this classification problem, though there has been considerable progress recently.

The proofs of the finiteness theorems in [2] and [22] are very different from one another. The method of [22] goes back to [21] and is based on a careful analysis of the local combinatorial structure of the Coxeter polyhedron of a reflection group, whereas the proof in [2] takes advantage of the global arithmetic and geometric properties of reflection groups. Both methods are effective in principle but the quantitative bounds which one can extract from the proofs are extremely large and have no practical value. Improvement of these bounds has been the subject of subsequent work. Currently, the best known general results are obtained through a combination of the two aforementioned methods (see however [5] for a different approach in a restricted setting). This paper is not an exception; we will first obtain new quantitative bounds in dimension $n = 3$ based on the method of Agol [1] (related to [2]), and then combine them with Nikulin's results from [22] and [23] to produce the best current bounds in higher dimensions.

Belolipetsky partially supported by a CNPq research grant.

Associated to an arithmetic group is a field k called its field of definition. In our case the field k is a totally real algebraic number field. The degree n_k and the discriminant d_k of this field are the basic invariants of the arithmetic reflection groups we are interested in. Finiteness of arithmetic maximal hyperbolic reflection groups in dimension 3 was first proven by Agol [1], but his argument as it stands does not provide quantitative bounds for their invariants. This issue was addressed in [4], where explicit bounds for the degrees and discriminants of the fields of definition were obtained. In this paper we use more careful number theoretic considerations which allow us to significantly improve the results in [4]. Thus, in Theorem 3.1 we show that the degree $n_k \leq 9$ (compare with $n_k \leq 35$ in [4]). We also give explicit upper bounds for the discriminant d_k in each of the possible degrees (see Theorem 4.9). These results bring the invariants of the defining fields close to the range of existing tables [29]. We also present a heuristic argument which allows us to further narrow the list (Theorem 4.12). The proofs of the main theorems use Borel's volume formula [6] and results of Chinburg and Friedman [8] (as in [4]) together with a Laplace-eigenvalue bound of Luo, Rudnick and Sarnak [16], a refinement of the Odlyzko discriminant bounds that takes into account primes of small norm [25], [7], and Louboutin's improvement of the Brauer–Siegel theorem [14]. We also note that several aspects of our proof were inspired by the work of Doyle, Linowitz and Voight [9] on isospectral but not isometric arithmetic 2- and 3-manifolds of small volume. The details are given in Sections 3 and 4. In the last section we show how these results may be combined with the previous work of Maclachlan and Nikulin in order to deduce a general upper bound $n_k \leq 25$ which is valid for all dimensions.

Acknowledgements. We would like to thank John Voight for helpful conversations.

2. PRELIMINARIES

The group of orientation preserving isometries of the hyperbolic 3-space $\mathrm{Isom}^+(\mathbb{H}^3)$ is isomorphic to $\mathrm{PGL}(2,\mathbb{C})$. In this section we will recall the definition of arithmetic subgroups of $\mathrm{PGL}(2,\mathbb{C})$. We refer to [6] and [18] for further details and examples of arithmetic subgroups.

Let K be a number field with exactly one complex place, $\mathcal{O}_K$ its ring of integers, and B a quaternion algebra over K. Let $\mathcal{D}$ be a maximal $\mathcal{O}_K$-order of B, denote by $\mathcal{D}^1$ its group of elements of norm 1 and let $\mathrm{Norm}(\mathcal{D}^1)$ be the normalizer of $\mathcal{D}^1$ in B. Consider a K-embedding $\rho : B \hookrightarrow \mathrm{M}(2,\mathbb{C})$ associated with the complex place of K. The projection

$$\Gamma_{\mathcal{D}} = P\rho(\mathrm{Norm}(\mathcal{D}^1)) < \mathrm{PGL}(2,\mathbb{C}),$$

where $P : \mathrm{M}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C})$, is then a discrete finite covolume subgroup of $\mathrm{PGL}(2,\mathbb{C})$. Any subgroup of $\mathrm{PGL}(2,\mathbb{C})$ which is commensurable with some such group $\Gamma_{\mathcal{D}}$ is called an *arithmetic subgroup*, and the field K is called its *field of definition*.

A discrete subgroup of $\mathrm{PGL}(2,\mathbb{C})$ is called *maximal* if it is maximal within its commensurability class with respect to inclusion. It can be shown that the groups $\Gamma_{\mathcal{D}}$ described above are in fact maximal arithmetic subgroups of $\mathrm{PGL}(2,\mathbb{C})$. These are not the only maximal arithmetic subgroups; by Borel's theorem the commensurability class of an arithmetic subgroup contains infinitely many maximal elements which can be described explicitly [6]. For the purpose of this article, however, the representatives $\Gamma_{\mathcal{D}}$ will suffice.

In order to deal with hyperbolic reflection groups we need to consider more general *Kleinian groups*, which are the discrete subgroups of the full group of isometries $\mathrm{Isom}(\mathbb{H}^3)$. All the above cited material can be applied to a reflection group Γ by considering its index 2 orientation preserving subgroup Γ^+, for which we have $\Gamma^+ < \mathrm{PGL}(2,\mathbb{C})$. With regards to arithmeticity it should

be pointed out that arithmetic reflection groups fall into a special class of *arithmetic subgroups defined by quadratic forms* [28, Lemma 7]. It follows that in this case the defining field K is of *even degree* and has a totally real subfield k such that $[K : k] = 2$ (cf. [18, Theorem 10.4.1]). The latter are the fields of definition which were discussed in the introduction. In order to stress the difference between K and k we will always refer to the latter as the *totally real field of definition* of an arithmetic (reflection) group.

3. Main theorem

In this section we shall prove our main result:

Theorem 3.1. *Let K be the field of definition of an arithmetic Kleinian reflection group. Then the degree of K is at most 18.*

Corollary 3.2. *The degree of the totally real field of definition of an arithmetic hyperbolic reflection group in dimension 3 is at most 9.*

Borel's volume formula shows that the group $\Gamma_{\mathcal{D}}$ has minimal volume within its commensurability class [6, Theorem 5.3]. It therefore suffices to exhibit an upper bound V for the covolume of a maximal arithmetic Kleinian group which may contain a reflection group and show that if $\Gamma_{\mathcal{D}}$ is a maximal arithmetic Kleinian group which has covolume less than V and whose field of definition K is of even degree, then K has degree at most 18. Agol [1] has shown that one may take $V = 128\pi^2$. In his derivation of this bound Agol makes critical use of the Burger–Sarnak–Vignéras lower bound of $3/4$ for the minimal non-zero eigenvalue of the Laplacian on $\mathbb{H}^3/\Gamma$. This bound has subsequently been improved by a number of authors, allowing us to obtain a slightly stronger volume bound.

Proposition 3.3. *The volume of a maximal arithmetic Kleinian group which contains a reflection group is at most $108\pi^2$.*

Proof. Let Γ be a maximal arithmetic Kleinian group containing a reflection subgroup. In his proof [1, Theorem 6.1] that the covolume of Γ is at most $128\pi^2$, Agol showed that

$$\lambda_1(\mathbb{H}^3/\Gamma)\left(1/2 \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma)\right)^{2/3} \leq 3(8\pi^2)^{2/3},$$

where $\lambda_1(\mathbb{H}^3/\Gamma)$ is the minimal non-zero eigenvalue of the Laplacian on $\mathbb{H}^3/\Gamma$.

Note that a maximal arithmetic Kleinian group is congruence [1, Lemma 5.1] (see also [13, Lemma 4.2]). The proposition now follows from Luo, Rudnick and Sarnak's proof [16] that $\lambda_1(\mathbb{H}^3/\Gamma) \geq 21/25$ whenever Γ is a congruence subgroup of an orthogonal group defined by a quadratic form over a totally real number field. □

Remark 3.4. We remark that the generalized Ramanujan conjecture implies that the minimal non-zero eigenvalue of the Laplacian of a congruence arithmetic Kleinian group is greater than or equal to 1. This would lead to a volume bound of $84\pi^2$.

Fix a maximal arithmetic Kleinian group $\Gamma_{\mathcal{D}}$ with defining field K of even degree n and defining quaternion algebra B. It is well known that such a field K must contain a unique complex place. Denote by $\mathrm{Ram}_f(B)$ the set of finite primes of K which ramify in B. By Borel's volume formula [6], we have (cf. [8, Proposition 2.1]):

$$\text{(3.5)} \qquad \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathcal{D}}) \geq \frac{8\pi^2\zeta_K(2)d_K^{3/2}[\mathcal{O}_K^\times : \mathcal{O}_{K,+}^\times]\prod_{\mathfrak{p}\in\mathrm{Ram}_f(B)}\left(\frac{N(\mathfrak{p})-1}{2}\right)}{(8\pi^2)^{n_K}h(K,2,B)}.$$

Here ζ_K denotes the Dedekind zeta function of K, d_K the absolute value of the discriminant of K, $\mathcal{O}_K^\times$ the units of K, $\mathcal{O}_{K,+}^\times$ the totally positive units of K, n_K the degree of K, and $h(K,2,B)$ the order of the (wide) ideal class group of K modulo the squares of all classes and the classes corresponding to primes in $\mathrm{Ram}_f(B)$.

Denote by m the rank (over $\mathbb{F}_2$) of the group of totally positive units of K modulo squares so that $[\mathcal{O}_K^\times : \mathcal{O}_{K,+}^\times] = 2^{n_K-1-m}$. Let $\omega_2(B)$ denote the number of primes of K that have norm 2 and which ramify in B. Recall that $\zeta_K(s)$ has an Euler product expansion $\zeta_K(s) = \prod_{\mathfrak{p}}(1 - N(\mathfrak{p})^{-s})^{-1}$ which converges for $\mathfrak{Re}(s) > 1$. It follows that $\zeta_K(2) \geq (4/3)^{\omega_2(B)}$. If $\mathfrak{p}$ is a prime of K which ramifies in B, then $\frac{N(\mathfrak{p})-1}{2} \geq 1$ unless $N(\mathfrak{p}) = 2$. It follows that $\prod_{\mathfrak{p}\in\mathrm{Ram}_f(B)}\left(\frac{N(\mathfrak{p})-1}{2}\right) \geq (1/2)^{\omega_2(B)}$.

Our proof of Theorem 3.1 will follow from a careful analysis of the number theoretic quantities present in (3.5). All of our calculations were performed with the open-source computer algebra system Sage [27].

We begin by combining Proposition 3.3 with (3.5) to deduce

$$\text{(3.6)} \qquad 108\pi^2 \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathcal{D}}) \geq \frac{8\pi^2(2/3)^{\omega_2(B)}2^{n_K-1-m}d_K^{3/2}}{(8\pi^2)^{n_K}h(K,2,B)}.$$

We wish to derive an upper bound for the root discriminant $\delta_K := d_K^{1/n_K}$ of K. To do so we must first bound the value of $h(K,2,B)$. Lemma 4.3 of [8] implies

$$\text{(3.7)} \qquad 108\pi^2 \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathcal{D}}) > 0.69\exp\left(0.37n_K - \frac{19.08}{h(K,2,B)}\right).$$

If $n_K \geq 20$ then equation (3.7) implies that $h(K,2,B) \leq 333$. Because $h(K,2,B)$ is a power of 2, we see that $h(K,2,B) \leq 256$. By employing the trivial bounds $\omega_2(B) \leq n_K$ and $m \leq n_K - 1$ (the latter follows from Dirichlet's unit theorem) we see from (3.6) that

$$\text{(3.8)} \qquad \delta_K \leq 24.117 \cdot 229^{1/n_K}.$$

If $n_K \geq 38$ then the discriminant bounds of Odlyzko [24] (see also Martinet [20]) imply that $\delta_K \geq 28.730$. This contradicts (3.8), hence $n_K \leq 36$.

We now show that $n_K \neq 20$. The proof that $n_K \notin \{22, 24, 26, 28, 30, 32, 34, 36\}$ is similar.

Proposition 3.9. *No maximal arithmetic Kleinian group with volume less than $108\pi^2$ has a defining field K of degree 20.*

Proof. Suppose that a maximal arithmetic Kleinian group with volume less than $108\pi^2$ has field of definition K of degree 20. Employing the bound $h(K,2,B) \leq 256$ along with the trivial bounds $\omega_2(B) \leq n_K$ and $m \leq n_K - 1$ in (3.6) shows that $\delta_K \leq 31.646$. We now employ a refinement of the Odlyzko bounds that takes into account the existence of primes of small norm, due to Poitou [25] and further developed by Brueggeman and Doud [7]. The Odlyzko-Poitou bounds show that

the root discriminant of a number field of degree 20 with 18 real places and at least 6 primes of norm 2 is at least 33.387. This is a contradiction and allows us to deduce that $\omega_2(B) \leq 5$. We now return to (3.6) to obtain a smaller upper bound for δ_K and then again utilize the Odlyzko–Poitou discriminant bounds in order to obtain a better bound for $\omega_2(B)$. Repeating this process shows that $\omega_2(B) = 2$ and $\delta_K \leq 24.810$.

We claim that the class number h_K of K is one. If $h_K \geq 2$ then the Hilbert class field of K is a number field of degree $20 \cdot h_K \geq 40$ with $18 \cdot h_K \geq 36$ real places and with root discriminant δ_K. Applying the Odlyzko bounds to this field shows that $\delta_K \geq 27.950$. This is a contradiction and proves our claim.

We now employ a theorem of Armitage and Fröhlich [3] in order to deduce that $\lfloor n_K/2 \rfloor \geq m$. Equation (3.6) now implies that $\delta_K < 16.751$ for all $\omega_2(B) \leq 2$. The Odlyzko bounds imply that no field of degree 20 with 18 real places has root discriminant less than 19.365. This contradiction finishes our proof. □

4. Discriminant bounds and class numbers

In this section we deduce explicit upper bounds for the root discriminant of the defining field K of an arithmetic Kleinian reflection group. The case in which the degree n_K of K is equal to 2 was handled completely in [4], hence we assume $4 \leq n_K \leq 18$.

In order to obtain reasonable discriminant bounds we must first obtain an upper bound for $h(K, 2, B)$. We note that (3.7) no longer provides us with such a bound, as the inequality always holds for $n_K \leq 18$. Instead, we will make use of Louboutin's refinement of the Brauer–Siegel theorem [14].

Let κ_K denote the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K. By the analytic class number formula [12, Chapter XIII] we have:

$$\kappa_K = \frac{(2\pi) h_K \operatorname{Reg}_K 2^{n_K - 2}}{w_K d_K^{1/2}}, \tag{4.1}$$

where h_K is the class number of K, Reg_K the regulator of K and w_K the number of roots of unity contained in K. Note that our assumption that $n_K > 2$ implies that there exists an embedding of K into $\mathbb{R}$, hence $w_K = 2$.

As $h(K, 2, B) \leq h_K$, we may derive an upper bound for $h(K, 2, B)$ by combining an upper bound for κ_K with a lower bound for Reg_K.

Louboutin (see [14, Prop. 2] and [15, Thm. 1]) has shown that

$$\kappa_K \leq \left(\frac{e \log(d_K)}{2(n_K - 1)} \right)^{n_K - 1} \tag{4.2}$$

holds for all K, and that when K is a quartic field with root discriminant $\delta_K \geq 17$ one has

$$\kappa_K \leq \frac{\log^{n_K - 1}(d_K)}{2^{n_K - 1} (n_K - 1)!}. \tag{4.3}$$

We note that in order to apply [15, Thm. 1], which implies (4.3), it is necessary that the quotient of zeta functions $\zeta_K(s)/\zeta(s)$ be entire. This is well known when $n_K = 4$.

Following the work of Zimmert [30], Friedman [10, pages 620-621] has shown that

$$\operatorname{Reg}_K \geq 0.0062 e^{(0.241 n_K + 0.497(n_K - 2))}, \tag{4.4}$$

and that $\mathrm{Reg}_K \geq 0.36$ when $n_K = 4$ and $\mathrm{Reg}_K \geq 1.23$ when $n_K = 6$. Combining (4.1), (4.2) and (4.4) yields:

$$h(K,2,B) \leq \frac{d_K^{1/2}\,(e\log(d_K))^{n_K-1}}{0.0062\pi 2^{2n_K-3}\,(n_K-1)^{n_K-1}\,e^{(0.241n_K+0.497(n_K-2))}}. \tag{4.5}$$

When $n_K = 4$ we get the improved bound

$$h(K,2,B) \leq \frac{d_K^{1/2}\log^{n_K-1}(d_K)}{0.36\pi 2^{2n_K-3}\,(n_K-1)!}, \tag{4.6}$$

and when $n_K = 6$ we get

$$h(K,2,B) \leq \frac{d_K^{1/2}\,(e\log(d_K))^{n_K-1}}{1.23\pi 2^{2n_K-3}\,(n_K-1)^{n_K-1}}. \tag{4.7}$$

It is clear that there are only finitely many values of d_K for which both (3.5) and (4.5) (or (4.6), (4.7) if $n_K = 4, 6$) can hold. In the table below we list, for every degree $4 \leq n_K \leq 18$, the associated upper bound for δ_K.

(4.8)

n_K	δ_K
4	< 668
6	649
8	639
10	503
12	445
14	395
16	361
18	346

As the root discriminant of any subfield of K does not exceed δ_K, we may summarize the above discussion as follows.

Theorem 4.9. *The root discriminant of the totally real field of definition k of an arithmetic hyperbolic reflection group in dimension 3 is bounded above by the entry corresponding to $2n_k$ in (4.8).*

Using the computer algebra system Sage, one may easily enumerate totally real fields of small degree and discriminant. The following corollaries are immediate consequences of Theorem 4.9.

Corollary 4.10. *There are at most $135,643$ real quadratic fields which are the totally real field of definition of an arithmetic hyperbolic reflection group in dimension 3.*

Corollary 4.11. *There are at most $17,449,721$ real cubic fields which are the totally real field of definition of an arithmetic hyperbolic reflection group in dimension 3.*

While the root discriminant bounds in (4.8) represent a major step towards the enumeration of all totally real defining fields of arithmetic Kleinian reflection groups (compared for instance, with the discriminant bounds produced in [4]), they are still in general much too large for the fields to appear in existing tables of totally real number fields [29]. The primary reason that the discriminant bounds in (4.8) are as large as they are is that because we lack a good bound for $h(K,2,B)$, we are forced to bound it by the full class number, which we in turn bound using Louboutin's refinement

of the Brauer–Siegel theorem [14]. It seems likely that these bounds are far from optimal and that every field satisfying our root discriminant bounds has 2-class number considerably smaller than the bounds produced by refinements of the Brauer–Siegel theorem.

Recall that if G is a finite abelian group and p is a prime number, then the *p-rank* of G is the number of factors of p-power order which occur in the direct sum decomposition of G into cyclic primary components. We conclude this section by classifying all totally real defining fields of arithmetic Kleinian reflection groups whose defining fields K have class group with 2-rank at most 12. These fields have associated $h(K, 2, B)$ values sufficiently small that the methods used in the proof of Proposition 3.9 can be employed in order to show the degree of such a field is at most 6 and produce root discriminant bounds which allow for their enumeration. Specifically, we get the following theorem.

Theorem 4.12. *Let K be the defining field of an arithmetic Kleinian reflection group Γ and assume that the 2-rank of the class group of K is at most 12. Then the totally real defining field of Γ has degree at most 6 and appears in Voight's table of totally real number fields* [29].

We also include a table of the root discriminant bounds produced for each even degree $4 \leq n_K \leq 12$ and remind the reader that for each degree n_K, any totally real subfield of K will also have root discriminant less than δ_K.

(4.13)

n_K	δ_K
4	149
6	82
8	60
10	29
12	16

Remark 4.14. We briefly state an interesting consequence of the failure of the hypothesis of Theorem 4.12. Consider for instance, the defining field K of an arithmetic hyperbolic Kleinian reflection group with class group having 2-rank greater than 12. By Theorem 3.1, the degree n_K of K is at most 18. It now follows from the theorem of Golod–Shafarevich [11] (see also [26, Theorem 3]) that K has an infinite 2-class field tower. Odlyzko [24] has shown that in this situation the root discriminant δ_K of K satisfies $\delta_K \geq (60.8395)^{(n_K-2)/n_K} \cdot (22.3816)^{(2/n_K)}$ unconditionally, and that $\delta_K \geq (215.3325)^{(n_K-2)/n_K} \cdot (44.7632)^{(2/n_K)}$ if one assumes the generalized Riemann hypothesis. These bounds can be combined with those of (4.8) in order to obtain a narrower interval in which δ_K can lie. It should be noted that the existence of number fields having a unique complex place, infinite 2-class field tower and root discriminants satisfying the upper bounds of (4.8) would be extremely significant. For instance, the smallest known (in terms of root discriminant) totally real field with infinite class field tower has root discriminant $1058.56...$ [19].

5. Degree bounds for higher dimensions

In this section we will discuss some higher dimensional implications of our main theorem.

Let Γ be an arithmetic hyperbolic reflection group in dimension n. For $n \neq 3$ the definition of arithmeticity given in Section 2 would not apply. Here we are going to use another approach which is more specific to reflection groups. We first introduce some necessary terms.

Consider a fundamental polyhedron $P \subset \mathbb{H}^n$ for a discrete group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ generated by reflections in the sides of P. Let $G(P) = [a_{ij}]_{i,j=1}^m$ denote the Gram matrix of P (cf. e.g. [18,

Section 10.4]). Define two fields $\tilde{k}(P)$ and $k(P)$ as follows:

$$\tilde{k}(P) = \mathbb{Q}(\{a_{ij} :\ i,j = 1,2,\dots,m\}),$$
$$k(P) = \mathbb{Q}(\{b_{i_1 i_2 \dots i_r}\}),$$

where for all subsets $\{i_1, i_2, \dots, i_r\} \subset \{1,2,\dots,m\}$ we define the cyclic products $b_{i_1 i_2 \dots i_r}$ by

$$b_{i_1 i_2 \dots i_r} = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_r i_1}.$$

Vinberg [28] proved that the group Γ is *arithmetic* if P has finite volume, all the cyclic products $b_{i_1 i_2 \dots i_r}$ are algebraic integers, the field $\tilde{k}(P)$ is totally real, and ${}^{\sigma}G(P) = [\sigma(a_{ij})]$ is positive semi-definite for all Galois embeddings $\sigma : \tilde{k}(P) \hookrightarrow \mathbb{C}$ such that $\sigma | k(P) = \mathrm{Id}$. It can be checked that when $n = 3$ the field $k(P)$ coincides with the totally real field of definition of Γ considered above.

Based on Vinberg's criterion Nikulin [21] introduced the notion of V-arithmeticity: a reflection group Γ is called *V-arithmetic* if P satisfies all the conditions of Vinberg's criterion except possibly the requirement that it has finite volume. This notion allows us to apply induction on the sides of the arithmetic hyperbolic polyhedra and to reduce certain questions to smaller dimension. It plays a central role in Nikulin's work on arithmetic reflection groups.

Another important ingredient of Nikulin's method is the notion of minimality. Given a positive real number t, the polyhedron P is called *t-minimal* if $a_{ij} < t$ for all entries a_{ij} of the Gram matrix $G(P)$.

The main result of [22] states that the degree of the totally real field of definition of an arithmetic hyperbolic reflection group does not exceed the maximum of the degrees of the possible fields in dimensions $n = 2, 3$ and the *transition constant* $N(14)$. The constant $N(14)$ is equal to the maximal degree of the totally real fields of definition of certain V-arithmetic groups with four generators whose fundamental polyhedra have minimality 14 (see [21, 23]). It was proven in [23] that $N(14)$ is bounded above by 25.

Our main theorem implies that $n_k \leq 9$ for dimension $n = 3$. Maclachlan proved in [17] that $n_k \leq 11$ for $n = 2$. Together with the above cited results of Nikulin these bounds imply:

Corollary 5.1. *The degree of the totally real fields of definition of arithmetic hyperbolic reflection groups in all dimensions is at most* 25.

Let us mention that before our work similar methods were used to show that the degree is bounded above by 35, and the critical place was in dimension 3. Our main result allows to shift the attention back to the value of the transition constant, where further improvements can be expected.

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